JOURNAL OF NUMBER THEORY 28, 66–68 (1988)

Every Integer is a Sum or Difference of 28 Integral Eighth Powers

L. N. VASERSTEIN

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

Communicated by H. Zassenhaus

Received July 15, 1986

The main result is stated as the title.

Let $v(k)$ be the least s such that every integer is the sum of s elements of the form $\pm z^k$, where z is an integer. In 1939, Fuchs and Wright [4] obtained some lower and upper bounds for $v(k)$ with $k \leqslant 20$. In particular, they proved that $16 \leqslant v(8) \leqslant 30$. Since that time a lot of work on $v(k)$ was done. The new bounds $9 \leqslant v(4) \leqslant 10$ [8] and $v(10) \leqslant 30$ [15] were obtained. Better upper bounds for $v(k)$ with $k \geqslant 11$ were computed (see, for example, [1, 2, 5, 9, 10, 11, 12]; for the history of the problem, see [6, 7, 13, 14, and Chaps. XXIV and XXV]).

But it seems there was no improvement of the bounds for $v(8)$. Here we prove the following theorem.

THEOREM 1. $17 \leqslant v(8) \leqslant 28$.

To obtain the lower bound 17, we use an idea of [8] (which is reproduced in [6]. This idea gives, in fact the following general result.

THEOREM 2. $2^{n+1}+1 \leqslant v(2^n)$ *for any integer* $n \geqslant 2$.

Indeed, modulo $2^{n+2}=4k$, every kth power z^k is 0 or 1. Let us show that the number $6k$ is not a sum or difference of $2k$ kth powers (hence $v(k) \geqslant 2k+1=2^{n+1}+1$). Otherwise, $6k=4k+2k$ is the sum of $2k$ integers and either each of them is 1 modulo $4k$ or each of them is -1 modulo $4k$. The second case is impossible because $6k$ is positive. The first case is also impossible because $2k<6k<3^k$.

CONJECTURE 3. $v(k) \leqslant 2k+1$ *for all* k.

0022-314X/88 $3.00

Remark 4. The only values $v(k)$ known exactly are $v(1)=1$ and $v(2)=3$.

To obtain the upper bound $v(8) \leqslant 28$, we find an identity similar to Rao's identity

$$\begin{aligned}(a^5c+bdx)^6 &+ (a^5d-bcx)^6+(b^5c-adx)^6+(b^5d+acx)^6 \\ &- (a^5c-bdx)^6-(a^5d+bcx)^6-(b^5c+adx)^6-(b^5d-acx)^6 \\ &= 12abcd(c^4-d^4)(a^{24}-b^{24})\,x. \qquad (5)\end{aligned}$$

This identity was used in [4] to prove that $v(6) \leqslant 14$. It is written in [4]: "We have not been able to find similar identities for higher values of k."

The desired identity for $k=8$ is

$$\begin{aligned}(a^{56}b^{31}c^{54}x+b^{31}c^{110})^8 &+ (a^{25}c^{116}x+a^{25}b^{88}c^{28})^8+(a^{25}b^{31}c^{85}x+a^{57}b^{63}c^{21})^8 \\ &- (a^{56}b^{31}c^{54}x-b^{31}c^{110})^8-(a^{25}c^{116}x-a^{25}b^{88}c^{28})^8 \\ &- (a^{25}b^{31}c^{85}x-a^{57}b^{63}c^{21})^8+(a^{55}b^{25}c^{61}x-a^7b^{73}c^{61})^8 \\ &+ (a^{20}bc^{120}x-a^{60}b^{81})^8+(a^{31}b^{36}c^{74}x-a^{15}b^{63}c^{63})^8 \\ &- (a^{55}b^{25}c^{61}x+a^7b^{73}c^{61})^8-(a^{20}bc^{120}x+a^{60}b^{81})^8 \\ &- (a^{31}b^{36}c^{74}x+a^{15}b^{63}c^{63})^8 \\ &= 16a^{56}b^{248}c^{120}ex \text{ with } e=c^{704}+a^{144}b^{368}c^{192}+a^{368}b^{224}c^{112} \\ &\quad - a^{48}b^{288}c^{368}-a^{384}b^{320}-a^{80}b^{229}c^{395}. \qquad (6)\end{aligned}$$

After a cancellation pattern that worked was found, the exponents above were found by solving a system of 9 linear equations with 9 unknowns.

The identity (6) shows that every term of a non-trivial artithmetic progression m_0x ($m_0 \neq 0$) is the sum or difference of 12 eighth powers (to obtain m_0, fix numbers a, b, c such that the right hand side of (6) does not vanish). By [4], $\Delta(8)=16$; i.e., every integer is the sum or difference of 16 eighth powers modulo any given $m \neq 0$ (we saw above that the bound 16 here is the best possible). Taking $m=m_0$, we obtain that $v(8) \leqslant 12+\Delta(8)=12+16=28$. The proof of Theorem 1 is completed.

The identity (6) also proves the following theorem.

THEOREM 7. *The form* $x_1^8+\cdots+x_6^8-x_7^8-\cdots-x_{12}^8$ *of degree* 8 *in* 12 *variables represents every element of any algebra* A *with* 1 *over the field of rational numbers.*

The best previously known result in degree 8 due to Essott (and later independently discovered by Phillips) was for the form

$x_1^8 + \cdots + x_7^8 - x_8^8 - \cdots - x_{14}^8$ in 14 variables. This was used in [4] to prove that $v(8) \leqslant 14 + \Delta(8) = 30$.

By [11], a lower bound for the number of variables in Theorem 7 (independent on A) is 3. When the center of A contains a primitive 64th root of 1, then the form $x_1^8 + \cdots + x_8^8$ in 8 variables represents everything in A (an easy exercise).

Remark 8. For the other two numbers related with the Waring problem in degree 8, it is known that $g(8) = 279$ and $32 \leqslant G(8) \leqslant 73$.

Remark 9. From our proof of Theorem 1 it is clear that there exist infinitely many representations of each integer as a sum or difference of 28 eight powers.

Open problem 10. It is unknown whether $v(k) \to \infty$ as $k \to \infty$ (by contrast, it is known that $G(k) \geqslant k+1$). Moreover I do not know whether the equation $x^k + y^k - z^k = -2$ has only finitely many integral solutions x, y, z, k with even positive $k \geqslant 4$.

References

1. D. P. Banerjee, On the solution of the "easier" Waring problem, *Bull. Calcutta Math. Soc.* **34** (1942), 197–199. MR.5.91.e.
2. I. Barrodale, A note on equal sums of like powers, *Math. Comput.* **20** (1966), 318–322.
3. L. E. Dickson, "History of Number Theory," Vol. 2.
4. W. H. J. Fuchs and E. M. Wright, The "easier" Waring problem, *Quart. J. Math. Oxford* **10** (1939), 190–209.
5. H. Gupta, A note on sums of powers, *Math. Student* **19** (1951).
6. G. H. Hardy and E. M. Wright, "An Introduction to the Theory of Numbers," 4th ed., Chap. XXI, Oxford Univ. Press, London/New York.
7. L. K. Hua, On Tarry's problem, *Quart. J. Math. Oxford* **9** (1938), 315–320.
8. W. Hunter, The representation of numbers by sums of fourth powers, *J. London Math. Soc.* **16** (1941), 177–179.
9. L. J. Lander, Three thirteens, *Math. Comput.* **27** (1973), 397. MR48#3878.
10. S. N. Mehrotra, On sums of powers, *Math. Student* **35** (1967), 73–77; **37** (1969), 204–205.
11. D. J. Newman and M. Slater, Waring's problem for the ring of polynomials, *J. Number Theory* **11**: 4 (1979), 477–487.
12. T. Rai, Easier Waring problem, *J. Sci. Res. Benares Hindu Univ.* **1** (1951), 5–19. MR.13.626f.
13. E. M. Wright, On Tarry's problem, *Quart. J. Math. Oxford*, **6** (1935), 261–267; **7** (1936), 43–45; **8** (1937), 48–50.
14. E. M. Wright, Equal sums of like powers, *Canad. Math. Bull.* **8** (1965), 193–202.
15. E. M. Wright, The Tarry-Escot and the "easier" Waring problems, *J. Reine Angew. Math.* **311/312** (1979), 170–173.